

Equivalent Norm - Let a linear space  $L$  be made into a normed linear space in two ways and let the two norms of a vector  $x$  in  $L$  be denoted by  $\|x\|_1$  and  $\|x\|_2$ . Then these norms are said to be equivalent, written  $\| \cdot \|_1 \sim \| \cdot \|_2$  iff they generate the same topology on  $L$ .

Theorem 1: Let  $N$  be a normed linear space and suppose two norms  $\| \cdot \|_1$  and  $\| \cdot \|_2$  are defined on  $N$ . Then these norms are equivalent if and only if there exist positive real numbers  $m$  and  $M$  such that

$$m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1$$

for every  $x$  in  $N$ .

Proof: - Let  $N_i$  be the normed linear space with the norm  $\| \cdot \|_i$  ( $i=1,2$ ). Let  $T(x) = x$  and consider  $T$  as a linear transformation with domain  $N_1$  and range  $N_2$ . Then  $T^{-1}$  is a linear transformation with domain  $N_2$  and range  $N_1$  such that

$$T(x) = x \Rightarrow T^{-1}(x) = x.$$

Now  $T$  is continuous  $\Rightarrow T$  is bounded

$\Rightarrow \exists$  positive number  $M$  such that

~~$$\|T(x)\|_2 \leq M\|x\|_1, \forall x \in N_1$$~~

$$\|T(x)\|_2 \leq M\|x\|_1, \forall x \in N_1$$

$$\Rightarrow \|x\|_2 \leq M\|x\|_1, \forall x \in N_1 \quad \text{--- (1)}$$

( $\because T(x) = x$ )

Again  $T^{-1}$  is continuous  $\Rightarrow T^{-1}$  is bounded

$\Rightarrow \exists$  positive number  $K$  such that

$$\|T^{-1}(x)\|_1 \leq K\|x\|_2, \forall x \in N_2.$$

$$\Rightarrow \|x\|_1 \leq K \|x\|_2 \quad (\because T^{-1}(x) = x)$$

$$\Rightarrow \frac{1}{K} \|x\|_1 \leq \|x\|_2 \quad (\because K > 0)$$

$$\Rightarrow m \|x\|_1 \leq \|x\|_2, \text{ setting } \frac{1}{K} = m \quad \text{--- (2)}$$

Also  $T$  and  $T^{-1}$  are continuous

$\Rightarrow$  inverse image of open sets in  $N_2$  and  $N_1$  under  $T$  and  $T^{-1}$  respectively are open in  $N_1$  and  $N_2$

$\Rightarrow$  open sets in  $N_1$  and  $N_2$  are the same

( $\because T$  and  $T^{-1}$  are identity transformations)

$\Rightarrow \| \cdot \|_1$  and  $\| \cdot \|_2$  induce the same topology on  $N$

From (1), (2), (3) and the above definition we conclude that  $\| \cdot \|_1$  and  $\| \cdot \|_2$  are equivalent.

$\Rightarrow$  there exist positive numbers  $m$  and  $M$  such that  $m \|x\|_1 \leq \|x\|_2 \leq M \|x\|_1, \forall x \in N$ .

~~Proof of further discussion of equivalent norms~~

**Theorem (B):** - Let  $N$  and  $N'$  be normed linear spaces and let  $T: N \rightarrow N'$  be any linear transformation. If  $N$  is finite-dimensional then  $T$  is continuous (or bounded).

**Proof:** - Let  $\dim N = n$  and  $\{e_1, e_2, \dots, e_n\}$  be a basis for  $N$ . Then to each  $x \in N$ , there exist unique scalars  $d_1, d_2, \dots, d_n$  such that

$$x = \sum_{i=1}^n d_i e_i$$

Since  $T$  is linear, we have

$$T(x) = \sum_{i=1}^n d_i T(e_i) \quad \text{--- (1)}$$

Now, consider zeroth norm on  $N$  defined by

$$\|x\|_0 = \max |d_i| \quad \text{--- (2)}$$

If  $\|\cdot\|$  is the norm on  $N'$ , then (1) gives

$$\begin{aligned} \|T(x)\| &= \left\| \sum_{i=1}^n \alpha_i e_i \right\| \leq \sum_{i=1}^n |\alpha_i| \|e_i\| \\ &\leq \|x\|_0 \sum_{i=1}^n \|e_i\| \quad \text{--- (3)} \end{aligned}$$

Since the basis is fixed  $\sum_{i=1}^n \|e_i\|$  is a positive constant.  
Hence letting  $M = \sum_{i=1}^n \|e_i\|$ , we have

$$\|T(x)\| \leq M \|x\|_0$$

Thus  $T$  is bounded and hence it is continuous.

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